

# SOME PHENOMENA IN TAUTOLOGICAL RINGS OF MANIFOLDS

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**ABSTRACT.** We prove several basic ring-theoretic results about tautological rings of manifolds  $W$ , that is, the rings of generalised Miller–Morita–Mumford classes for fibre bundles with fibre  $W$ . Firstly we provide conditions on the rational cohomology of  $W$  which ensure that its tautological ring is finitely-generated, and we show that these conditions cannot be completely relaxed by giving an example of a tautological ring which fails to be finitely-generated in quite a strong sense. Secondly, we provide conditions on torus actions on  $W$  which ensure that the rank of the torus gives a lower bound for the Krull dimension of the tautological ring of  $W$ . Lastly, we give extensive computations in the tautological rings of  $\mathbb{CP}^2$  and  $S^2 \times S^2$ .

## 1. INTRODUCTION

**1.1. Recollections on tautological rings.** A smooth fibre bundle  $\pi : E \rightarrow B$  with closed  $d$ -dimensional fibre  $W$  equipped with an orientation of the vertical tangent bundle  $T_\pi E$  has characteristic classes defined as follows. For each characteristic class  $c \in H^k(BSO(d))$  of oriented  $d$ -dimensional vector bundles, we may form

$$\kappa_c(\pi) := \int_\pi c(T_\pi E) \in H^{k-d}(B),$$

the *generalised Mumford–Morita–Miller class* (or  $\kappa$ -class) associated to  $c$ , by evaluating  $c$  on the vector bundle  $T_\pi E$  and integrating the result along the fibres of the map  $\pi$ . This construction may in particular be applied to the universal such fibre bundle, whose base space is the classifying space  $B\mathrm{Diff}^+(W)$  of the topological group of orientation-preserving diffeomorphisms of  $W$ , to give universal characteristic classes  $\kappa_c \in H^*(B\mathrm{Diff}^+(W))$ .

If we work in cohomology with rational coefficients then  $H^*(BSO(d); \mathbb{Q})$  is generated by the Pontrjagin and Euler classes, and in this case we define the *tautological ring*

$$R^*(W) \subset H^*(B\mathrm{Diff}^+(W); \mathbb{Q})$$

to be the subring generated by all classes  $\kappa_c$ . Our goal is to describe some quantitative and qualitative properties of these rings, for certain manifolds  $W$ .

Before doing so, we introduce some variants. The topological group  $\mathrm{Diff}^+(W, *)$  of diffeomorphisms of  $W$  which fix a marked point  $* \in W$  has a homomorphism to  $GL_d^+(\mathbb{R})$  by sending a diffeomorphism  $\varphi$  to its differential  $D\varphi_*$  at the marked point. On classifying spaces this gives a map

$$s : B\mathrm{Diff}^+(W, *) \longrightarrow BGL_d^+(\mathbb{R}) \simeq BSO(d)$$

and for each  $c \in H^*(BSO(d); \mathbb{Q})$  we may also form  $s^*c \in H^*(B\mathrm{Diff}^+(W, *); \mathbb{Q})$ . We let the *tautological ring fixing a point*  $R^*(W, *) \subset H^*(B\mathrm{Diff}^+(W, *); \mathbb{Q})$  be the subring generated by all the classes  $\kappa_c$  and  $s^*c$ .

Finally, if  $B\mathrm{Diff}^+(W, D^d)$  is the classifying space of the group of diffeomorphisms of  $W$  which are the identity near a marked disc  $D^d \subset W$ , then we let the *tautological*

ring fixing a disc  $R^*(W, D^d) \subset H^*(B\text{Diff}^+(W, D^d); \mathbb{Q})$  be the subring generated by all the classes  $\kappa_c$ . There are thus natural maps

$$R^*(W) \longrightarrow R^*(W, *) \longrightarrow R^*(W, D^d)$$

whose composition is surjective.

**1.2. Finiteness.** Our first result concerns conditions under which the rings  $R^*(W)$  and  $R^*(W, *)$  are suitably finite.

**Theorem A.** *Let  $W$  be a closed smooth oriented  $2n$ -manifold, and assume that either*

- (H1)  $H^*(W; \mathbb{Q})$  is non-zero only in even degrees, or
- (H2)  $H^*(W; \mathbb{Q})$  is non-zero only in degrees 0,  $2n$  and odd degrees, and  $\chi(W) \neq 0$ .

*Then*

- (i)  $R^*(W)$  is a finitely-generated  $\mathbb{Q}$ -algebra, and
- (ii)  $R^*(W, *)$  is a finitely-generated  $R^*(W)$ -module.

The result under hypothesis (H2) generalises a theorem of Grigoriev [Gri13], and proceeds by establishing the same basic source of relations among  $\kappa$ -classes found by Grigoriev. As the later results of [Gri13] and the results of [GGR15] are deduced almost entirely from this basic source of relations, the same results largely follow assuming only hypothesis (H2). For example, for  $g > 1$ ,  $k$  odd, and  $n \geq k$ , it follows that

$$\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \dots, \kappa_{ep_{n-1}}] \longrightarrow R^*(\#^g S^k \times S^{2n-k})/\sqrt{0}$$

is surjective, which was obtained in [GGR15] only in the case  $k = n$ . We give details of this in Corollary 4.1.

The result under hypothesis (H1) is entirely new and its method of proof is different. The proof establishes concrete relations among  $\kappa$ -classes, whose form is that of the Cayley–Hamilton trace identity for matrices. This is because the proof goes through an application of something like the Cayley–Hamilton identity, but in the category of parametrised  $H\mathbb{Q}$ -module spectra rather than the category of vector spaces. Later we shall describe some explicit calculations done using these relations.

**1.3. Krull dimension.** Our second main result is a general technique, continuing on from our work with Galatius and Grigoriev [GGR15, §4], for estimating the Krull dimension (for which we write  $\text{Kdim}$ ) of the rings  $R^*(W)$  from below in terms of torus actions on  $W$ . The general statement is Theorem 3.1, but the hypotheses of that theorem are somewhat involved: we state here one of its corollaries with hypotheses which are easy to verify.

**Corollary B.** *Let a  $k$ -torus  $T$  act effectively on  $W$ , and suppose that either*

- (i)  $\chi(W) \neq 0$  and the fixed set  $W^T$  is connected, or
- (ii) the fixed set  $W^T$  is discrete.

*Then  $\text{Kdim}(R^*(W)) \geq k$ .*

For example, if  $W^{2n}$  is a quasitoric manifold then case (ii) gives the estimate  $\text{Kdim}(R^*(W)) \geq n$ . As another example, if  $W = \#^g S^n \times S^n$  with  $n$  odd then it is a consequence of equivariant localisation that *any* torus action on  $W$  has connected fixed set, so by case (i) restricting the  $SO(n) \times SO(n)$ -action on  $W$  constructed in [GGR15, §4] to a maximal torus (which has rank  $n - 1$ ) we obtain  $\text{Kdim}(R^*(W)) \geq n - 1$  for  $g > 1$ , which fits with the calculation of that paper. This example admits many variants: the construction of [GGR15, §4] can be easily

modified to give a  $SO(k) \times SO(2n - k)$ -action on  $\#^g S^k \times S^{2n-k}$ , so for any odd  $k$  and any  $n$  we have

$$\text{Kdim}(R^*(\#^g S^k \times S^{2n-k})) \geq n - 1.$$

We shall say more about this example in Corollary 4.1.

**1.4. Examples.** In the last section of the paper we demonstrate several phenomena in tautological rings by calculations for specific manifolds. The following result is complementary to Theorem A, and shows that the hypotheses of that theorem cannot be completely removed.

**Theorem C.** *There are closed smooth manifolds  $W$  for which  $R^*(W)/\sqrt{0}$  is not finitely-generated as a  $\mathbb{Q}$ -algebra. There are examples of any dimension  $4k + 2 \geq 6$ , and in dimensions  $4k + 2 \geq 14$  such manifolds can also be assumed to be simply-connected.*

To show the effectiveness of the relations between  $\kappa$ -classes which we found in the proof of Theorem A, we apply them to the simplest manifold whose tautological ring is not yet known, namely  $\mathbb{CP}^2$ . These relations, along with relations associated to the Hirzebruch  $\mathcal{L}$ -classes coming from index theory, give the following.

**Theorem D.** *The ring  $R^*(\mathbb{CP}^2)$  has Krull dimension 2. The ring  $R^*(\mathbb{CP}^2, D^4)$  has dimension at most 7 over  $\mathbb{Q}$ .*

In fact, we show that the ring  $R^*(\mathbb{CP}^2)/\sqrt{0}$  is equal to either

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/(4\kappa_{p_1^2} - 7\kappa_{ep_1}) \cap (\kappa_{p_1^2} - 2\kappa_{ep_1}, 316\kappa_{ep_1}^3 - 343\kappa_{p_1^4}),$$

whose variety is the union of a line and a plane, or

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/(4\kappa_{p_1^2} - 7\kappa_{ep_1}),$$

whose variety is a plane. It would be interesting to determine which case occurs, and very interesting if it is the first case.

Finally, we give a calculation which shows that the lower bound of Corollary B is not always sharp. The 3-torus cannot act effectively on  $S^2 \times S^2$ , and yet

**Theorem E.** *The ring  $R^*(S^2 \times S^2)$  has Krull dimension 3 or 4.*

The lower bound on the Krull dimension comes from a 1-parameter family of 2-torus actions, to which the method of proof of Corollary B is applied. The upper bound comes from the relations between  $\kappa$ -classes which we found in the proof of Theorem A.

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## 2. TAUTOLOGICAL RELATIONS AND FINITE GENERATION

*Unless specified, all cohomology in this paper will be taken with  $\mathbb{Q}$  coefficients.*

In this section we describe some techniques for obtaining relations between tautological classes, which for some manifolds  $W$  suffice to establish that  $R^*(W)$  is finitely-generated. The techniques we will introduce are perhaps more important than any particular application that can be made, but Theorem A will be a consequence.

One consequence of conclusion (ii) of Theorem A is that  $R^*(W, *)$  is integral over  $R^*(W)$ . In fact, this integrality statement implies the two finiteness statements, as follows.

**Lemma 2.1.** *Suppose that  $W$  is a closed smooth oriented  $d$ -manifold such that  $R^*(W, *)$  is integral over  $R^*(W)$ . Then*

- (i)  $R^*(W)$  is a finitely-generated  $\mathbb{Q}$ -algebra, and
- (ii)  $R^*(W, *)$  is a finitely-generated  $R^*(W)$ -module.

*Proof.* As  $H^*(BSO(d))$  is a finitely-generated  $\mathbb{Q}$ -algebra, we know *a priori* that  $R^*(W, *)$  is finitely-generated as an algebra over  $R^*(W)$ , so under the integrality assumption it follows that  $R^*(W, *)$  is in fact finitely-generated as a module over  $R^*(W)$ .

For  $c \in H^*(BSO(d))$ , let  $p_c(z) = a_0 + a_1 z + \cdots + z^k \in R^*(W)[z]$  be a monic polynomial which is satisfied by  $s^*c \in R^*(W, *)$ . For any other  $x \in H^*(BSO(d))$ , multiplying  $p_c(s^*c) = 0 \in R^*(W, *)$  by  $s^*x$  and fibre integrating shows that

$$\kappa_{c^k x} \in \mathbb{Q}[a_0, a_1, \dots, a_{k-1}, \kappa_x, \kappa_{xc}, \dots, \kappa_{xc^{k-1}}] \subset R^*(W)$$

which, along with the fact that  $H^*(BSO(d))$  is finitely-generated, implies that  $R^*(W)$  is finitely-generated.  $\square$

Thus in order to prove Theorem A we shall actually show that  $R^*(W, *)$  is integral over  $R^*(W)$ .

**2.1. Outline.** To motivate the proof of Theorem A let us first explain its proof under hypothesis (H1) and an additional assumption: that the universal smooth oriented fibre bundle  $W \rightarrow E \xrightarrow{\pi} B = B\text{Diff}^+(W)$  satisfies the Leray–Hirsch property in rational cohomology, i.e. that  $\pi_1(B)$  acts trivially on  $H^*(W)$  and the Serre spectral sequence for  $\pi : E \rightarrow B$  collapses. (The proof of Theorem A under hypothesis (H1) is a technical device which allows the following argument to be made without this additional assumption.)

We first develop some algebra. Under this assumption,  $H^*(E)$  is a free finitely-generated  $H^*(B)$ -module, say with generators  $\bar{x}_1, \dots, \bar{x}_k \in H^*(E)$  lifting a basis for  $H^*(W)$ . Furthermore, as  $W$  has all its cohomology in even degrees,  $H^{ev}(E)$  is a free finitely-generated module over the commutative ring  $H^{ev}(B)$ , with generators the  $\bar{x}_i$ . Any nontrivial  $c \in H^*(BSO(2n))$  has even degree, so  $c = c(T_\pi E) \in H^{ev}(E)$ . Now

$$- \cdot c : H^{ev}(E) \longrightarrow H^{ev}(E)$$

is a  $H^{ev}(B)$ -module map, so has a characteristic polynomial  $\chi_c(z) \in H^{ev}(B)[z]$ , and by the Cayley–Hamilton theorem (for finite modules over a commutative ring, alias the determinantal trick) we have  $\chi_c(c) = 0 \in H^{ev}(E)$ . Furthermore, the coefficients of the characteristic polynomial  $\chi_c(z)$  may be expressed as polynomials in the elements

$$\text{Tr}(- \cdot c^i : H^{ev}(E) \rightarrow H^{ev}(E)) \in H^{ev}(B),$$

which make sense as  $H^{ev}(E)$  is a finite free  $H^{ev}(B)$ -module and hence dualisable.

We now pass to topology. Recall that the Becker–Gottlieb transfer homomorphism  $\text{trf}_\pi^* : H^*(E) \rightarrow H^*(B)$  may be expressed, when  $\pi : E \rightarrow B$  is an oriented smooth fibre bundle, as  $\pi_!(e(T_\pi E) \cdot -)$ . In particular for  $c \in H^*(BSO(2n))$  we have that  $\text{trf}_\pi^*(c(T_\pi E)) = \kappa_{ec}$  is a tautological class. The description of the Becker–Gottlieb transfer in e.g. [DP80] shows that when  $H^*(E)$  is a dualisable  $H^*(B)$ -module (for example finite and free, as is the case here) then

$$\text{trf}_\pi^*(x) = \text{Tr}(- \cdot x : H^*(E) \rightarrow H^*(E)) \in H^*(B),$$

that is, it is the sum over  $i$  of the coefficients  $a_{i,i} \in H^*(B)$  obtained when we express

$$\bar{x}_i \cdot x = \sum_j a_{i,j} \cdot \bar{x}_j.$$

Now when  $x$  has even degree then all  $a_{i,j}$  have even degree, and so this trace is the same as the trace of  $-\cdot x : H^{ev}(E) \rightarrow H^{ev}(E)$ .

The consequence of this is that the polynomial  $\chi_c(z)$  has its coefficients in the subring generated by the  $\text{trf}_\pi^*(c^i)$ , and as these are tautological classes we deduce that  $s^*c$  is integral over  $R^*(W)$  and hence that  $R^*(W, *)$  is integral over  $R^*(W)$ . Theorem A in the case we are considering follows by applying Lemma 2.1.

**2.2. Proof of Theorem A.** Let  $W \rightarrow E \xrightarrow{\pi} B$  be a smooth oriented fibre bundle over a finite CW-complex base, and let  $(\text{Sp}_{/B}, \wedge_B, \mathbb{S}_B^0)$  denote the symmetric monoidal category of parametrised spectra over  $B$ . The fibre bundle  $\pi : E \rightarrow B$  defines a parameterised spectrum  $\Sigma_B^\infty E \in \text{Sp}_{/B}$ , the constant parameterised spectrum  $H\mathbb{Q}$  defines a ring object in  $\text{Sp}_{/B}$ , and we may consider the function object  $C := F(\Sigma_B^\infty E, H\mathbb{Q})$ . This is an  $H\mathbb{Q}$ -module object in  $\text{Sp}_{/B}$ .

Let  $(\mathcal{C}, \otimes, \mathbb{1})$  denote the homotopy category of parametrised  $H\mathbb{Q}$ -module spectra over  $B$ , with fibrewise smash product as the symmetric monoidal structure and unit  $\mathbb{1} = H\mathbb{Q}$ ; the function object  $C$  is then an object of  $\mathcal{C}$ . The category  $\mathcal{C}$  is a (graded)  $\mathbb{Q}$ -linear  $\otimes$ -category, and hence for any object  $X \in \mathcal{C}$  we have a map of rings

$$e : \mathbb{Q}[\Sigma_n] \longrightarrow [X^{\otimes n}, X^{\otimes n}].$$

For each partition  $\lambda \vdash n$  we have the (central) Schur idempotent

$$d_\lambda := \frac{\dim S^\lambda}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \sigma \in \mathbb{Q}[\Sigma_n],$$

where  $S^\lambda$  is the Specht module associated to  $\lambda$  and  $\chi_\lambda$  is its character. Thus  $e(d_\lambda)$  is an idempotent endomorphism of  $X^{\otimes n}$ , and we write  $S_\lambda(X)$  for the corresponding retract of  $X^{\otimes n}$ . We in particular write  $\wedge^n X := S_{(1^n)}(X)$  and  $\text{Sym}^n(X) := S_{(n)}(X)$ .

Let  $(\mathcal{V}, \otimes, \mathbb{Q}[0])$  denote the (graded)  $\mathbb{Q}$ -linear  $\otimes$ -category of graded rational vector spaces. The same construction may be applied to define  $S_\lambda(V_*)$  for any graded vector space  $V_*$ .

**Lemma 2.2.** *Let  $X \in \mathcal{C}$  be an  $H\mathbb{Q}$ -module spectrum parametrised over  $B$  such that for each fibre  $X_b \in H\mathbb{Q}\text{-mod}$  we have  $S_\lambda(\pi_*(X_b)) = 0$ . Then  $S_\lambda(X) \simeq *$ .*

*Proof.* We must compute the graded vector space  $[\mathbb{1}, S_\lambda(X)]_*$  of maps in  $\mathcal{C}$  and show it is zero. This is the same as computing  $[\mathbb{S}_B^0, S_\lambda(X)]_*$  in the homotopy category of parametrised spectra over  $B$ . For this purpose there is a strongly convergent spectral sequence (cf. [MS06, Theorem 20.4.1])

$$H_p(B; [\mathbb{S}^0, S_\lambda(X)_b]_q) \Rightarrow [\mathbb{S}_B^0, S_\lambda(X)]_{p+q}$$

where  $[\mathbb{S}^0, S_\lambda(X)_b]_q$  denotes the local coefficient system over  $B$  which at  $b \in B$  consists of the  $q$ th homotopy group of the fibre  $S_\lambda(X)_b$  of the parametrised spectrum  $S_\lambda(X)$  over  $b$ . The functor

$$(-)_b : \mathcal{C} \longrightarrow \text{Ho}(H\mathbb{Q}\text{-mod})$$

which extracts the fibre at  $b \in B$ , and the functor

$$\pi_*(-) : \text{Ho}(H\mathbb{Q}\text{-mod}) \longrightarrow \mathcal{V}$$

which takes homotopy groups, are both monoidal and preserve (homotopy) colimits. Hence they preserve the functor  $S_\lambda(-)$  on each of the three categories involved. By the first functor we have  $S_\lambda(X)_b \simeq S_\lambda(X_b)$ . By the second functor we have  $\pi_*(S_\lambda(X_b)) \cong S_\lambda(\pi_*(X_b))$  which is zero by assumption. Hence  $[\mathbb{S}^0, S_\lambda(X)_b]_q = 0$  for all  $q \in \mathbb{Z}$  so the spectral sequence is identically zero. It follows that  $S_\lambda(X)$  is contractible.  $\square$

**2.2.1. Schur-finiteness and trace identities.** Deligne has introduced the notion of *Schur-finiteness* of an object  $X$  in a  $\mathbb{Q}$ -linear  $\otimes$ -category to be the property that  $S_\lambda(X)$  is trivial for some partition  $\lambda \vdash n$ . In this section we consider this notion applied to the category  $(\mathcal{C}, \otimes, \mathbb{1})$  and so consider  $X$  an  $H\mathbb{Q}$ -module spectrum parametrised over  $B$  such that  $S_\lambda(X) \simeq *$ , and let us in addition suppose that  $X$  is dualisable in  $\mathcal{C}$ . Let us write  $D(X)$  for the dual of  $X$ , with duality structure

$$\eta : \mathbb{1} \longrightarrow X \otimes D(X) \quad \varepsilon : D(X) \otimes X \longrightarrow \mathbb{1}.$$

Given an endomorphism  $f : X \rightarrow \Sigma^p X$ , we may form the endomorphism

$$X^{\otimes n} \xrightarrow{X \otimes f^{\otimes n-1}} \Sigma^{p(n-1)} X^{\otimes n} \xrightarrow{d_\lambda} \Sigma^{p(n-1)} X^{\otimes n}$$

and take the trace over all but the first copy of  $X$ . The endomorphism of  $X$  so obtained is nullhomotopic because the idempotent  $d_\lambda : X^{\otimes n} \rightarrow X^{\otimes n}$  factors through  $S_\lambda(X)$  which is contractible by assumption.

We now translate this into formulas. We have  $d_\lambda = \frac{\dim S^\lambda}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \sigma$  so the essential calculation is to describe the endomorphism of  $X$  obtained from  $\sigma \circ (X \otimes f^{\otimes n-1}) : X^{\otimes n} \rightarrow X^{\otimes n}$  by taking the trace over all but the first copy of  $X$ . This is a universal construction in  $\mathbb{Q}$ -linear  $\otimes$ -categories, and has been worked out by del Padrone [Pad06, Proposition 2.2.4] following André-Kahn [AK02, Proposition 7.2.6], giving the identity

$$(2.1) \quad 0 = \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \text{Tr}(f^{\circ l(\gamma_2)}) \cdots \text{Tr}(f^{\circ l(\gamma_q(\sigma))}) \cdot f^{\circ l(\gamma_1)-1} \in [X, \Sigma^{p(n-1)} X]$$

where  $\sigma = \gamma_1 \cdot \gamma_2 \cdots \gamma_q(\sigma)$  is a decomposition into disjoint cycles, with 1 being in the support of  $\gamma_1$ , and  $l(\gamma_i)$  denotes the length of the cycle  $\gamma_i$ .

**2.2.2. Proof of Theorem A under the first hypothesis.** Under hypothesis (H1) the cohomology  $H^*(W)$  is concentrated in even degrees, and let us suppose that it has total dimension  $k$ . Then we have  $\wedge^{k+1}(H^*(W)) = 0$  and so, as  $\pi_*(C_b) = H^{-*}(E_b) \cong H^{-*}(W)$  for each  $b \in B$ , it follows from Lemma 2.2 that  $\wedge^{k+1} C \simeq *$ .

The following is the appropriate form of Poincaré duality in our setting.

**Lemma 2.3.**  *$C$  is a dualisable object of  $\mathcal{C}$ , with dual  $\Sigma^{2n} C$ .*

*Proof.* This will follow from the parameterised form of Atiyah duality, along with the Thom isomorphism. As  $B$  is a finite CW-complex, we may choose an embedding  $E \subset B \times \mathbb{R}^N$  and identify a tubular neighbourhood of it with the fibrewise normal bundle  $p : \nu \rightarrow E$  of  $E$  in  $B \times \mathbb{R}^N$ . Writing  $\text{Th}_B(\nu)$  for the fibrewise 1-point compactification of  $\nu$  over  $B$ , there is a map

$$\phi : B \times S^N \longrightarrow \text{Th}_B(\nu) \xrightarrow{d} \text{Th}_B(\nu) \wedge_B E_+$$

where the first map collapses the complement of  $\nu$  in each fibre, and the second map is given by sending  $(b, x)$  to  $((b, x), p(b, x))$  on  $\nu \subset \text{Th}_B(\nu)$ . Passing to parametrised spectra, let  $D := F(\Sigma_B^{\infty-N} \text{Th}_B(\nu), H\mathbb{Q})$ . On one hand the map

$$D \otimes C \xrightarrow{\text{product}} F(\Sigma_B^{\infty-N} \text{Th}_B(\nu) \wedge_B \Sigma_B^\infty E, H\mathbb{Q}) \xrightarrow{\phi^*} F(\mathbb{S}_B^0, H\mathbb{Q}) = \mathbb{1}$$

exhibits  $C$  and  $D$  as dual in  $\mathcal{C}$  (this can be checked by restricting to fibres over  $B$ ). On the other hand the bundle  $\nu$  is oriented by assumption so there is a Thom class  $u \in H^{N-2n}(\text{Th}(\nu)) = [\Sigma_B^{\infty-N} \text{Th}_B(\nu), \Sigma^{-2n} H\mathbb{Q}]$ , for which the map

$$\begin{aligned} \Sigma^{2n} C &= F(\mathbb{S}_B^{-2n} \wedge_B \Sigma_B^\infty E, H\mathbb{Q}) \xrightarrow{H\mathbb{Q} \wedge^-} F(\Sigma^{-2n} H\mathbb{Q} \wedge_B \Sigma_B^\infty E, H\mathbb{Q} \wedge_B H\mathbb{Q}) \\ &\xrightarrow{-\circ u \wedge^E} F(\Sigma_B^{\infty-N} \text{Th}_B(\nu) \wedge_B \Sigma_B^\infty E, H\mathbb{Q} \wedge_B H\mathbb{Q}) \xrightarrow{d^*} F(\Sigma_B^{\infty-N} \text{Th}_B(\nu), H\mathbb{Q}) = D \end{aligned}$$

is an equivalence (this can be checked by restricting to fibres over  $B$ ).  $\square$

As  $C$  is dualisable in  $\mathcal{C}$  the discussion of the previous section applies, and, as  $C$  is a ring object in  $\mathcal{C}$ , multiplication by a class  $x \in H^p(E) = [\Sigma^{-p}\mathbb{1}, C]$  gives an endomorphism  $\hat{x} : C \rightarrow \Sigma^p C$ . The trace identity in this case, because  $\lambda = (1^{k+1})$  corresponds to the sign representation, is

$$(2.2) \quad 0 = \sum_{\sigma \in \Sigma_{k+1}} \text{sign}(\sigma) \cdot \text{Tr}(\hat{x}^{\text{ol}(\gamma_2)}) \cdots \text{Tr}(\hat{x}^{\text{ol}(\gamma_q(\sigma))}) \cdot \hat{x}^{\text{ol}(\gamma_1)-1} \in [C, \Sigma^{pk}C].$$

**Corollary 2.4.** *The polynomial*

$$\rho_x(z) := \frac{(-1)^k}{k!} \sum_{\sigma \in \Sigma_{k+1}} \text{sign}(\sigma) \cdot \text{trf}_\pi^*(x^{\text{l}(\gamma_2)}) \cdots \text{trf}_\pi^*(x^{\text{l}(\gamma_q(\sigma))}) \cdot z^{\text{l}(\gamma_1)-1} \in H^*(B)[z]$$

is monic of degree  $k$  and annihilates  $x \in H^*(E)$

*Proof.* The class  $\text{Tr}(\hat{x}^{\circ i}) \in [\mathbb{1}, \mathbb{1}]_* = H^{-*}(B)$  is  $\text{trf}_\pi^*(x^i)$  by the trace description of the Becker–Gottlieb transfer. Applying the map (2.2) to the unit  $\iota : \mathbb{1} \rightarrow C$  therefore gives

$$0 = \sum_{\sigma \in \Sigma_{k+1}} \text{sign}(\sigma) \cdot \text{trf}_\pi^*(x^{\text{l}(\gamma_2)}) \cdots \text{trf}_\pi^*(x^{\text{l}(\gamma_q(\sigma))}) \cdot x^{\text{l}(\gamma_1)-1} \in H^*(E).$$

The coefficient of  $x^k$  is the sum over the  $k!$ -many  $(k+1)$ -cycles  $\sigma \in \Sigma_{k+1}$  of  $\text{sign}(\sigma) = (-1)^k$ , which is  $k!(-1)^k$ . Thus after dividing by this coefficient we see that the monic polynomial  $\rho_x(z)$  is satisfied by  $x$ .  $\square$

Taking  $c \in H^*(BSO(2n))$  and using that  $\text{trf}_\pi^*(c^i) = \kappa_{c^i e}(\pi)$ , one obtains a monic polynomial  $\rho_c(z) \in R^*(W)[z]$  such that the element  $\rho_c(s^*c) \in R^*(W, *)$  is zero when evaluated on any smooth oriented fibre bundle with section over a finite CW-complex base space. But as a rational cohomology class is zero if and only if it vanishes when evaluated against every rational homology class, and a homology class may always be supported on a finite CW-complex, this implies that  $\rho_c(s^*c) = 0 \in R^*(W, *)$  and hence that  $R^*(W, *)$  is integral over  $R^*(W)$ . Theorem A under hypothesis (H1) follows by applying Lemma 2.1.

**2.2.3. Proof of Theorem A under the second hypothesis.** We shall first prove the following generalisation of a theorem of Grigoriev [Gri13].

**Theorem 2.5.** *Let  $W$  be a manifold of dimension  $2n$  having rational cohomology only in degrees  $0, 2n$ , and odd degrees, and let  $d := \dim_{\mathbb{Q}} H^{\text{odd}}(W)$ . Let  $\pi : E \rightarrow B$  be a smooth oriented fibre bundle with fibre  $W$ . Let  $a, b \in H^*(E)$  satisfy  $\pi_!(a) = \pi_!(b) = 0$ , and  $a$  have even degree. Then*

$$\pi_!(a^2)^{\lceil \frac{d+1}{2} \rceil} = 0 \quad \text{and} \quad \pi_!(ab)^{d+1} = 0.$$

Fibre integration  $\pi_! : H^*(E) \rightarrow H^{*-2n}(B)$  is realised in the category  $\mathcal{C}$  by the morphism  $\pi_! : C \rightarrow \Sigma^{-2n}\mathbb{1}$  dual to the unit  $\iota : \mathbb{1} \rightarrow C$ , using the self-duality of  $C$  described in Lemma 2.3. We may thus define an object  $D'$  by the fibre sequence

$$D' \longrightarrow C \xrightarrow{\pi_!} \Sigma^{-2n}\mathbb{1}.$$

The composition  $\mathbb{1} \xrightarrow{\iota} C \xrightarrow{\pi_!} \Sigma^{-2n}\mathbb{1}$  is nullhomotopic, as it lies in  $[\mathbb{1}, \Sigma^{-2n}\mathbb{1}] = H^{-2n}(B) = 0$ , so  $\iota$  lifts to a map  $\iota' : \mathbb{1} \rightarrow D'$  and we can define an object  $D$  by the cofibre sequence

$$\mathbb{1} \xrightarrow{\iota'} D' \longrightarrow D.$$

**Lemma 2.6.** *If  $W$  only has rational cohomology in degree  $0, 2n$ , and odd degrees, then  $\pi_*(D_b) \cong H^{\text{odd}}(W)$ , so if  $d := \dim_{\mathbb{Q}} H^{\text{odd}}(W)$  then  $\text{Sym}^{d+1}(D) \simeq *$ .*

*Proof.* The  $H\mathbb{Q}$ -module spectrum  $D_b$  is obtained by forming the fibre sequence  $D'_b \rightarrow F(W, H\mathbb{Q}) \xrightarrow{\pi_!} \Sigma^{-2n} H\mathbb{Q}$  and then the cofibre sequence  $H\mathbb{Q} \xrightarrow{\iota'_b} D'_b \rightarrow D_b$ . Now  $\pi_*(F(W, H\mathbb{Q})) = H^{-*}(W)$ , the map

$$(\pi_!)_* : \pi_{-2n}(F(W, H\mathbb{Q})) = H^{2n}(W) \longrightarrow \pi_{-2n}(\Sigma^{-2n} H\mathbb{Q}) = \mathbb{Q}$$

realises capping with the fundamental class so is surjective, and the map

$$(\iota'_b)_* : \mathbb{Q} = \pi_0(H\mathbb{Q}) \longrightarrow \pi_0(D'_b) \cong H^0(W)$$

realises the unit so is injective. It follows that  $\pi_*(D_b)$  is isomorphic to  $H^{-*}(W)$  with the classes in degrees 0 and  $2n$  removed. The rest follows from Lemma 2.2.  $\square$

The map

$$D' \otimes D' \longrightarrow C \otimes C \xrightarrow{\mu} C \xrightarrow{\pi_!} \Sigma^{2n} \mathbb{1}$$

is nullhomotopic when precomposed with  $D' \otimes \mathbb{1} \xrightarrow{D' \otimes \iota'} D' \otimes D'$  or  $\mathbb{1} \otimes D' \xrightarrow{\iota' \otimes D'} D' \otimes D'$ , so taking cofibres of these two maps gives a morphism

$$\phi : D \otimes D \longrightarrow \Sigma^{2n} \mathbb{1}.$$

*Proof of Theorem 2.5.* If  $a \in H^k(E) = [\Sigma^k \mathbb{1}, C]$  is such that  $\pi_!(a) = 0$  then it lifts to a map to  $D'$  and hence determines a map  $\bar{a} : \Sigma^k \mathbb{1} \rightarrow D$ . Similarly if  $b \in H^\ell(E)$  satisfies  $\pi_!(b) = 0$  then it gives a  $\bar{b} : \Sigma^\ell \mathbb{1} \rightarrow D$ . The class  $\pi_!(a \cdot b)$  may therefore be represented by

$$\Sigma^k \mathbb{1} \otimes \Sigma^\ell \mathbb{1} \xrightarrow{\bar{a} \otimes \bar{b}} D \otimes D \xrightarrow{\phi} \Sigma^{2n} \mathbb{1}.$$

Hence the class  $\pi_!(a \cdot b)^N$  may be written as

$$(\Sigma^k \mathbb{1})^{\otimes N} \otimes (\Sigma^\ell \mathbb{1})^{\otimes N} \xrightarrow{\bar{a}^N \otimes \bar{b}^N} D^{\otimes N} \otimes D^{\otimes N} \xrightarrow{\phi^N} (\Sigma^{2n} \mathbb{1})^{\otimes N}$$

but if  $k$  is even then  $\bar{a}^N : (\Sigma^k \mathbb{1})^{\otimes N} \rightarrow D^{\otimes N}$  factors through  $\text{Sym}^N(D)$  which is contractible as long as  $N \geq d + 1$ . Hence  $\pi_!(a \cdot b)^{d+1} = 0$ .

Similarly, if  $a = b$  then we can choose  $\bar{b} = \bar{a} : \Sigma^k \mathbb{1} \rightarrow D$  in which case the map  $\bar{a}^N \otimes \bar{a}^N : (\Sigma^k \mathbb{1})^{\otimes N} \otimes (\Sigma^k \mathbb{1})^{\otimes N} \rightarrow D^{\otimes N} \otimes D^{\otimes N}$  factors through  $\text{Sym}^{2N}(D)$ , which is contractible as long as  $2N \geq d + 1$ . Hence  $\pi_!(a^2)^{\lceil \frac{d+1}{2} \rceil} = 0$ .  $\square$

Now that we have Theorem 2.5, the entirety of Section 5 of [Gri13] goes through with only notational changes, as this only uses the statement of Grigoriev's theorem. In particular, for  $p \in H^*(BSO(2n))$  of even degree and  $\chi = \chi(W) \neq 0$ , the analogue of [Gri13, Example 5.19] gives the relation

$$\left( p - \frac{\kappa_{ep}}{\chi} - \frac{e\kappa_p}{\chi} + \frac{\kappa_{e^2\kappa_p}}{\chi^2} \right)^{d+1} = 0 \in R^*(W, *).$$

From this it is clear that  $R^*(W, *)$  is a finite  $R^*(W)$ -module, as the monomials in  $\mathbb{Q}[p_1, p_2, \dots, p_{n-1}, e]$  where no variable occurs with exponent larger than  $d$  give a generating set. Thus  $R^*(W, *)$  is integral over  $R^*(W)$ , so by Lemma 2.1 the algebra  $R^*(W)$  is finitely-generated. This proves Theorem A under hypothesis (H2).

**2.3. Tautological relations.** Under either hypothesis we have established more than Theorem A, as we have produced explicit relations in  $R^*(W, *)$ . Under hypothesis (H2) these relations are equal to those obtained by Grigoriev, and under hypothesis (H1) they are given by Corollary 2.4 as

$$0 = \sum_{\sigma \in \Sigma_{k+1}} \text{sign}(\sigma) \cdot \kappa_{ec^{l(\gamma_2)}} \cdots \kappa_{ec^{l(\gamma_q(\sigma))}} \cdot c^{l(\gamma_1)-1} \in R^*(W, *)$$

for each  $c \in H^*(BSO(2n))$ , where  $k = \dim_{\mathbb{Q}} H^*(W)$ . These may of course be pushed forward to obtain relations in  $R^*(W)$ .



More generally, the trace identity technique of Section 2.2.1 may be used to find relations among tautological classes for *any* manifold. Recall that given a fibre bundle  $W \rightarrow E \xrightarrow{\pi} B$  we have formed an associated object  $C \in \mathcal{C}$ . Let us write  $d_{ev} = \dim_{\mathbb{Q}} H^{ev}(W)$  and  $d_{odd} = \dim_{\mathbb{Q}} H^{odd}(W)$ . The first ingredient is the following consequence of a calculation of Deligne.

**Lemma 2.7.** *If  $\lambda$  is a partition whose Young diagram contains the rectangle  $(d_{ev} + 1) \times (d_{odd} + 1)$ , then  $S_{\lambda}(C) \simeq *$ .*

*Proof.* By Lemma 2.2 it is enough to verify that  $S_{\lambda}(\pi_*(C_b)) = 0$  for all  $b \in B$ . But  $\pi_*(C_b) \cong H^{-*}(W)$  and by [Del02, Corollary 1.9] a  $(\mathbb{Z}/2)$ -graded vector space is annihilated by  $S_{\lambda}(-)$  under the given assumption on its (super)dimension.  $\square$

In particular, for a given manifold  $W$  we may take  $\lambda$  to be the partition of  $n = (d_{ev} + 1) \cdot (d_{odd} + 1)$  with Young diagram equal to the rectangle  $(d_{ev} + 1) \times (d_{odd} + 1)$ , so that we have  $S_{\lambda}(C) \simeq *$  and hence by (2.1) we have the relation

$$0 = \sum_{\sigma \in \Sigma_n} \chi_{\lambda}(\sigma) \cdot \kappa_{ec^{l(\gamma_2)}} \cdots \kappa_{ec^{l(\gamma_q(\sigma))}} \cdot c^{l(\gamma_1)-1} \in R^*(W, *).$$

It is a simple exercise with the Murnaghan–Nakayama rule to show that the character  $\chi_{\lambda}$  vanishes on all  $n$ -cycles if both  $d_{ev} > 0$  and  $d_{odd} > 0$ . As  $d_{ev}$  cannot be zero, because  $\dim_{\mathbb{Q}} H^0(W) \neq 0$ , it follows that this relation gives a monic polynomial satisfied by  $c$  if and only if  $d_{odd} = 0$ . (This accounts for why we restricted to manifolds with only even rational cohomology in the first case of Theorem A.)

### 3. TORUS ACTIONS

In this section we suppose that we have a smooth action of the torus  $T = (S^1)^k$  on a  $d$ -dimensional orientable manifold  $W$ . We write  $W^T$  for the fixed set of this action. The Borel construction gives a smooth fibre bundle

$$(3.1) \quad W \longrightarrow W//T \xrightarrow{\pi} BT,$$

and following the usual notation of equivariant cohomology we write

$$H_T^* = H^*(BT; \mathbb{Q}) = \mathbb{Q}[x_1, x_2, \dots, x_k] \quad \text{and} \quad H_T^*(W) = H^*(W//T; \mathbb{Q}).$$

As (3.1) is a smooth fibre bundle, there is a ring homomorphism  $\rho : R^*(W) \rightarrow H_T^*$ , and we denote by  $R_T^* \leq H_T^*$  its image. Pulling back  $\pi$  along itself gives a smooth fibre bundle over  $W//T$  with canonical section, and so a ring homomorphism  $\rho_* : R^*(W, *) \rightarrow H_T^*(W)$ , and we denote by  $R_T^*(*) \leq H_T^*(W)$  its image.

Our goal in this section is to describe conditions on the manifold  $W$  and the action of  $T$  on  $W$  which allow us to deduce that  $\text{Kdim}(R^*(W)) \geq k$ . Our most general result is as follows.

**Theorem 3.1.** *Let  $V_1, V_2, \dots, V_p$  be an enumeration of the  $T$ -representations arising as normal spaces to points on  $W^T$ , and let  $B_i$  denote the Euler characteristic of the subspace of  $W^T$  consisting of those path components having normal representation  $V_i$ .*

*If some  $y_i \in \mathbb{Q}[y_1, y_2, \dots, y_p]$  is integral over the subring generated by*

$$\sum_{i=1}^p B_i y_i^n, \quad n = 1, 2, 3, \dots,$$

*then  $\text{Kdim}(R^*(W)) \geq k$ .*

It is perhaps not clear when the hypothesis of this theorem is likely to hold. The following lemma, which we learnt from [BCES16], gives a simple criterion.

**Lemma 3.2.** *Suppose that we have discarded the  $B_i$  which are zero, and that this is not all of them. If the remaining numbers  $B_1, B_2, \dots, B_p$  have all partial sums non-zero, then  $\mathbb{Q}[y_1, y_2, \dots, y_p]$  is finite over the subring generated by*

$$(3.2) \quad \sum_{i=1}^p B_i y_i^n, \quad n = 1, 2, 3, \dots,$$

and so every  $y_i$  is integral over this subring.

*Proof.* Write  $B \leq \mathbb{Q}[y_1, y_2, \dots, y_p]$  for the subring generated by the  $\sum_{i=1}^p B_i y_i^n$ , and  $B^+$  for the subset of positive-degree elements.

**Claim.** If  $\sqrt{(B^+)} = (y_1, y_2, \dots, y_p)$  then  $\mathbb{Q}[y_1, \dots, y_p]$  is finite over  $B$ .

Our proof of this claim follows the discussion at [Jef]. Under the assumption the quotient ring  $\mathbb{Q}[y_1, y_2, \dots, y_p]/(B^+)$  has every  $y_i$  nilpotent, so is a finite  $\mathbb{Q}$ -module; let  $z_1, z_2, \dots, z_m \in \mathbb{Q}[y_1, y_2, \dots, y_p]$  be lifts of these finitely-many generators, which can be taken to be homogeneous as the ideal  $(B^+)$  is homogeneous. We claim that these generate  $\mathbb{Q}[y_1, y_2, \dots, y_p]$  as a  $B$ -module; let  $M \subset \mathbb{Q}[y_1, y_2, \dots, y_p]$  be the  $B$ -submodule that they generate.

As the  $z_i$  are homogeneous, and  $B$  is generated by homogeneous elements,  $M$  is a graded submodule of  $\mathbb{Q}[y_1, y_2, \dots, y_p]$  with the monomial-length grading. Suppose  $p \in \mathbb{Q}[y_1, y_2, \dots, y_p]$  is an element of minimal grading which does not lie in  $M$ . Then we may write

$$p = \sum U_n z_n + \sum V_n \left( \sum_{i=1}^p B_i y_i^n \right)$$

with  $U_n \in \mathbb{Q}$  and  $V_n \in \mathbb{Q}[y_1, y_2, \dots, y_p]$ . But the  $V_n$  have strictly smaller degree than  $p$ , so lie in  $M$ , and hence  $p$  does too, which proves the claim.

In order to prove the lemma we must therefore show that  $(y_1, y_2, \dots, y_p) = (0, 0, \dots, 0)$  is the only simultaneous solution to the equations  $\sum_{i=1}^p B_i y_i^n = 0$  for  $n \in \mathbb{N}$ . If  $(y_1, y_2, \dots, y_p) \in \mathbb{Q}^p$  is a solution, then grouping like terms together we obtain *distinct* rational numbers  $\bar{y}_i$  solving the equations

$$\sum_{i=1}^q \bar{B}_i \bar{y}_i^n = 0$$

where each  $\bar{B}_i$  is a partial sum of the  $B_i$ , and hence non-zero by assumption. But this means that the vector  $(\bar{B}_1 \bar{y}_1, \dots, \bar{B}_q \bar{y}_q)$  is in the kernel of the Vandermonde matrix associated to  $(\bar{y}_1, \dots, \bar{y}_q)$ , so as the  $\bar{y}_i$  are all distinct,  $(\bar{B}_1 \bar{y}_1, \dots, \bar{B}_q \bar{y}_q) = 0$ , and as the  $\bar{B}_i$  are all non-zero,  $\bar{y}_i = 0$  as required.  $\square$

The following corollary, whilst not so powerful as Theorem 3.1, is often easier to apply as one does not need to classify the normal representations at the fixed set.

**Corollary 3.3.** *Let the path components  $X_1, X_2, \dots, X_\ell$  of the fixed set  $W^T$  have Euler characteristics  $A_1, A_2, \dots, A_\ell$ . If some  $x_i \in \mathbb{Q}[x_1, x_2, \dots, x_\ell]$  is integral over the subring generated by*

$$\sum_{i=1}^{\ell} A_i x_i^n, \quad n = 1, 2, 3, \dots,$$

then  $\text{Kdim}(R^*(M)) \geq k$ .

*Proof.* Consider the ring homomorphism  $\phi : \mathbb{Q}[x_1, x_2, \dots, x_\ell] \rightarrow \mathbb{Q}[y_1, y_2, \dots, y_p]$  defined by sending  $x_i$  to  $y_j$  if the normal representation at every point of  $X_i$  is  $V_j$ .

Then

$$\phi \left( \sum_{i=1}^{\ell} A_i x_i^n \right) = \sum_{i=1}^{\ell} A_i \phi(x_i)^n = \sum_{j=1}^p B_j y_j^n$$

so  $\phi$  sends the subring  $A \subset \mathbb{Q}[x_1, x_2, \dots, x_{\ell}]$  generated by the  $\sum_{i=1}^{\ell} A_i x_i^n$  onto the subring  $B \subset \mathbb{Q}[y_1, y_2, \dots, y_p]$  generated by the  $\sum_{i=1}^p B_i y_i^n$ .

If  $x_i$  is integral over  $A$  then there is a polynomial  $q(x) = \sum a_i x^i$  with coefficients in  $A$  such that  $q(x_i) = 0$ . Then  $q'(y) = \sum \phi(a_i) y^i$  is a polynomial over  $B$  such that  $q'(\phi(x_i)) = 0$ , so  $y_j = \phi(x_i)$  is integral over  $B$ , and hence Theorem 3.1 applies.  $\square$

**Example 3.4.** There are several standard conditions which oblige a torus action on a manifold  $W$  to have connected fixed-set. For example

- (i) Let  $W$  have dimension  $2n$ , and suppose that all its cohomology apart from  $H^0(W; \mathbb{Q})$  and  $H^{2n}(W; \mathbb{Q})$  lies in odd degrees. Then  $W^T$  is connected (by localisation in equivariant cohomology).
- (ii) If  $W$  has trivial even-dimensional rational homotopy groups, then  $W^T$  is empty or connected [Hsi75, Theorem IV.5].

In such cases  $\chi(W^T) = \chi(W)$ , so if this is non-zero then the hypotheses of Corollary 3.3 are satisfied.

**Example 3.5.** Suppose that the action of  $T^k$  on  $M$  has isolated fixed points, or more generally that all  $A_i$  are equal and non-zero. Then the subring generated by the  $\sum_{i=1}^{\ell} A_i x_i^n$  is the subring of symmetric polynomials in  $\mathbb{Q}[x_1, x_2, \dots, x_{\ell}]$ , and every  $x_i$  is integral over this so the hypotheses of Corollary 3.3 are satisfied.

This implies, for example, that if  $M^{2n}$  is a quasitoric manifold (that is, the “toric manifolds” of [DJ91]) then  $\text{Kdim}(R^*(M)) \geq n$ , and if  $G/K$  is a homogeneous space of rank zero then  $\text{Kdim}(R^*(G/K)) \geq \text{rank}(G)$ .

We now prepare for the proof of Theorem 3.1. Let  $X_1, X_2, \dots, X_{\ell}$  be the components of the fixed set  $W^T$ , with  $\dim(X_i) = d_i$ , let  $\nu_{X_i \subset W}$  be the normal bundle of  $X_i$ , and let  $\nu_i$  be the  $T^k$ -representation which arises as each fibre of  $\nu_{X_i \subset W}$ . Let us write  $A_i = \chi(X_i)$ , and write  $V_1, V_2, \dots, V_p$  for an enumeration of the representations  $\nu_i$  which arise. Then we have  $B_i = \sum_{j \text{ s.t. } \nu_j = \nu_i} A_j$ .

Let us write  $\rho_i : H_T^*(W) \rightarrow H_T^*(X_i)$  for the restriction map in equivariant cohomology, and  $\pi_i : H_T^*(W) \rightarrow H_T^{*-d}$  and  $(\pi_i)_! : H_T^*(X_i) \rightarrow H_T^{*-d_i}$  for the pushforward maps. Let  $S \subset H_T^*$  be the multiplicative subset of nonzero elements. By standard results in equivariant localisation theory,  $e(\nu_{X_i \subset W}) \in S^{-1}H_T^{d-d_i}(X_i)$  is a unit and we have a commutative diagram

$$\begin{array}{ccc} S^{-1}H_T^*(W) & \xrightarrow[\sim]{\bigoplus_i \frac{\rho_i}{e(\nu_{X_i \subset W})}} & \bigoplus_i S^{-1}H_T^{*+d_i-d}(X_i) \\ \downarrow \pi_i & & \downarrow \sum_i (\pi_i)_! \\ S^{-1}H_T^{*-d} & \xlongequal{\quad} & S^{-1}H_T^{*-d}. \end{array}$$

Using this diagram to compute  $\kappa_{ep_I} = \pi_!(e(TW)p_I(TW)) \in S^{-1}H_T^*$ , which we know lies in the subring  $H_T^*$ , gives

$$\begin{aligned} \kappa_{ep_I} &= \sum_{i=1}^{\ell} (\pi_i)_! \left( \frac{e(TX_i \oplus \nu_{X_i \subset W}) p_I(TX_i \oplus \nu_{X_i \subset W})}{e(\nu_{X_i \subset W})} \right) \\ &= \sum_{i=1}^{\ell} (\pi_i)_! (e(TX_i) p_I(TX_i \oplus \nu_{X_i \subset W})) \end{aligned}$$

and in  $H_T^*(X_i) = H_T^* \otimes H^*(X_i)$  we have

$$p_I(TX_i \oplus \nu_{X_i \subset W}) = p_I(\nu_i) \otimes 1 + \text{terms with a nontrivial } H^*(X_i) \text{ component.}$$

When we multiply by  $e(TX_i)$  and integrate over  $X_i$  we do not see the latter terms, so we get

$$(3.3) \quad \kappa_{ep_I} = \sum_{i=1}^{\ell} A_i p_I(\nu_i) \in H_T^*.$$

Grouping these terms by the representation types  $V_j$  instead gives

$$(3.4) \quad \kappa_{ep_I} = \sum_{i=1}^p B_i p_I(V_i) \in H_T^*.$$

*Proof of Theorem 3.1.* From (3.4) applied to  $p_I = p_j^n$  we find that

$$\sum_{i=1}^p B_i p_j(V_i)^n \in R_T^* \text{ for all } j \text{ and } n.$$

Applying the hypothesis of the theorem for each  $j$ , we find that there exists an  $i$  such that all  $p_j(V_i)$  lie in a common integral extension  $R_T^* \subseteq R' \subseteq H_T^*$ . On the one hand  $R'$  is integral over  $R_T^*$ . On the other hand by a theorem of Venkov [Ven59] the ring  $H_T^*$  is finite over the subring generated by the  $p_j(V_i)$  (as  $V_i$  is a faithful representation of  $T$ ), and hence is finite (and so integral) over  $R'$ . It follows that  $H_T^*$  is integral over  $R_T^*$ , so in particular they have the same Krull dimension, namely  $k$ .

Finally,  $R^*(W) \rightarrow R_T^*$  is surjective and so  $\text{Kdim}(R^*(W)) \geq \text{Kdim}(R_T^*) = k$ .  $\square$

The discussion so far gives a technique more general Theorem 3.1, but difficult to formalise in a single result. It is best described through an example.

**Proposition 3.6.** *Let  $T$  act on  $W$  with two fixed components  $X_1$  and  $X_2$ . Suppose that  $\chi(X_1) = -\chi(X_2) \neq 0$  but that the normal  $T$ -representations  $\nu_1$  and  $\nu_2$  at  $X_1$  and  $X_2$  have all Pontrjagin classes distinct (when they are non-zero). Then  $\text{Kdim}(R^*(W)) \geq k$ .*

*Proof.* We have that

$$\kappa_{ep_j^n} = p_j(\nu_1)^n - p_j(\nu_2)^n \in R_T^* \leq H_T^*$$

for all  $j$  and  $n$ , and  $p_j(\nu_1) - p_j(\nu_2) \neq 0 \in R_T^*$ . Hence

$$p_j(\nu_1) = \frac{1}{2} \left( p_j(\nu_1) - p_j(\nu_2) + \frac{p_j(\nu_1)^2 - p_j(\nu_2)^2}{p_j(\nu_1) - p_j(\nu_2)} \right) \in R_T^* [(p_j(\nu_1) - p_j(\nu_2))^{-1}].$$

Therefore after inverting the finite set

$$S := \{p_j(\nu_1) - p_j(\nu_2), j = 1, 2, \dots\}$$

of non-zero elements in  $R_T^* \leq H_T^* = \mathbb{Q}[x_1, x_2, \dots, x_k]$ , we find that the  $p_j(\nu_1)$  lie in  $S^{-1}R_T^*$ , and hence by Venkov's theorem that  $S^{-1}H_T^*$  is a finite  $S^{-1}R_T^*$ -module. As  $S^{-1}H_T^*$  still has Krull dimension  $k$  (its variety is the complement of finitely many hyperplanes in  $\mathbf{A}^k$ ), it follows that  $S^{-1}R_T^*$  has Krull dimension  $k$  and so  $\text{Kdim}(R_T^*) \geq k$ . Hence  $\text{Kdim}(R^*(W)) \geq k$ .  $\square$

## 4. EXAMPLES

**4.1. Manifolds with mostly odd cohomology.** Let  $W$  be a  $2n$ -dimensional manifold whose cohomology is only non-trivial in degrees 0,  $2n$ , and odd degrees, let  $k = \dim_{\mathbb{Q}} H^{\text{odd}}(W)$ , and suppose  $\chi(W) = 2 - k \neq 0$ . Then by Theorem A  $R^*(W)$  is finitely-generated and  $R^*(W, *)$  is a finite  $R^*(W)$ -module.

Furthermore, by our method of proof, Grigoriev's theorem holds for these manifolds (our Theorem 2.5). Therefore the results of Sections 2 and 3 of [GGR15] hold for  $W$  as well, as Grigoriev's theorem was the only external input. So if  $k > 2$  then

$$\mathbb{Q}[\kappa_{ep_1}, \dots, \kappa_{ep_{n-1}}] \longrightarrow R^*(W)/\sqrt{0}$$

is surjective. Hence  $\text{Kdim}(R^*(W)) \leq n - 1$ .

By Example 3.4 (i), if  $T = (S^1)^k$  acts on such a manifold  $W$  then the fixed set  $W^T$  is connected, so  $\text{Kdim}(R^*(W)) \geq k$ . The construction of [GGR15, Section 4.1] can be mimicked to obtain an action of  $SO(k) \times SO(2n - k)$  on  $\#^g S^k \times S^{2n-k}$  for any  $k$ , and the calculation of the characteristic classes  $\kappa_{ep_i}$  for the associated bundle is entirely analogous.

We obtain the following generalisation of the results of [GGR15].

**Corollary 4.1.** *For  $k$  odd and  $g > 1$  we have*

$$\mathbb{Q}[\kappa_{ep_1}, \dots, \kappa_{ep_{n-1}}] \xrightarrow{\sim} R^*(\#^g S^k \times S^{2n-k})/\sqrt{0}$$

and

$$R^*(\#^g S^k \times S^{2n-k})/\sqrt{0} \xrightarrow{\sim} R^*(\#^g S^k \times S^{2n-k}, *)/\sqrt{0}.$$

Furthermore  $(2 - 2g) \cdot c = \kappa_{ec} \in R^*(\#^g S^k \times S^{2n-k}, *)/\sqrt{0}$ , so

$$R^*(\#^g S^k \times S^{2n-k}, D^{2n})/\sqrt{0} = \mathbb{Q},$$

and hence  $R^*(\#^g S^k \times S^{2n-k}, D^{2n})$  is a finite-dimensional  $\mathbb{Q}$  vector space.

As in [GGR15] results can be obtained for  $g = 0$  or  $1$  too, but we shall not write them out here.

**4.2. Quasitoric manifolds.** A quasitoric manifold  $W^{2n}$  has by definition a smooth action of  $T = (S^1)^n$  with isolated fixed points, so by Corollary 3.3 has

$$\text{Kdim}(R^*(W)) \geq n.$$

Furthermore, the integral cohomology of  $W$  is supported in even degrees, so its rational cohomology is too, and therefore by Theorem A  $R^*(W)$  is finitely-generated and  $R^*(W, *)$  is a finite  $R^*(W)$ -module.

**4.3. Non-finite generation.** We shall give some examples of manifolds  $W$  for which  $R^*(W)$ , and in fact even  $R^*(W)/\sqrt{0}$ , is not finitely-generated. We shall do so by constructing actions of a torus  $T$  on  $W$  and showing that the tautological subring  $R_T^* \leq H_T^*$  is not finitely-generated. As  $H_T^*$  is an integral domain the natural surjection  $R^*(W) \rightarrow R_T^*$  factors through  $R^*(W)/\sqrt{0}$ , which therefore shows that  $R^*(W)/\sqrt{0}$  is not finitely-generated.

Before attempting this method there is an important observation to be made.

*Observation 4.2.* Let  $T = (S^1)^k$  act on  $W$  satisfying the hypotheses of Theorem 3.1; the proof of that theorem shows that the inclusion  $R_T^* \hookrightarrow H_T^*$  is integral.

As  $H_T^*$  is Noetherian, and  $H^*(BT; H^*(W))$  is a finitely-generated  $H_T^*$ -module, it follows from the Serre spectral sequence for the Borel construction that  $H_T^*(W)$  is a finitely-generated  $H_T^*$ -module and hence is integral over  $H_T^*$ .

Therefore the morphism  $R_T^* \rightarrow H_T^* \rightarrow H_T^*(W)$  is integral, so  $R_T^* \rightarrow R_T^*(*)$  is integral too. It then follows from the proof of Lemma 2.1 that  $R_T^* \rightarrow R_T^*(*)$  is finite and  $R_T^*$  is a finitely-generated  $\mathbb{Q}$ -algebra.

Therefore to pursue the programme we have suggested one should only try to use torus actions which *do not* satisfy the hypotheses of Theorem 3.1. The following allows us to construct manifolds with torus actions having prescribed normal representations and Euler characteristics of its fixed sets.

**Construction 4.3.** Fix a positive odd integer  $n$  and an even integer  $k$ . Let  $\Sigma(k)^{2n}$  be a manifold of Euler characteristic  $k$  obtained as  $\#^g S^n \times S^n$  (if  $k$  is non-positive) or  $\coprod^g S^{2n}$  (if  $k$  is positive). Let  $H(k)^{2n+1}$  be the manifold with boundary  $\Sigma(k)^{2n}$  given by  $\natural^g S^n \times D^{n+1}$  or  $\coprod^g D^{2n+1}$  respectively.

Let  $T$  be a torus, and suppose we are given even integers  $B_1, B_2, \dots, B_p$  and distinct faithful complex  $T$ -representations  $V_1, V_2, \dots, V_p$ , which are all of the same dimension and which have no trivial subrepresentations. Then we can form the  $T$ -manifolds

$$M(i) = M(B_i, V_i) := H(B_i)^{2n+1} \times \mathbb{S}(V_i) \cup_{\Sigma(B_i) \times \mathbb{S}(V_i)} \Sigma(B_i)^{2n} \times \mathbb{D}(V_i).$$

We may then let  $M$  be the disjoint union  $M = M(1) \sqcup M(2) \sqcup \dots \sqcup M(p)$ .

If one prefers a connected manifold, note that each path component of each  $M(i)$  contains a free orbit, and given a free orbit in two different path components we may cut out a neighbourhood of each free orbit and glue the  $T$ -manifolds together along the resulting boundaries. Doing this enough times yields a connected  $T$ -manifold with the same fixed-point data, and hence by localisation with the same characteristic classes.

**Lemma 4.4.** *The  $T$ -manifold  $M$  so obtained has  $\kappa_{p_I} = 0$  and*

$$\kappa_{ep_I} = \sum_{i=1}^p B_i \cdot p_I(V_i) \in H_T^*.$$

*Proof.* The second statement follows from (3.4). An analogous calculation shows that

$$\kappa_{p_I} = \sum_{i=1}^p (\pi_i)! \left( \frac{p_I(TX_i \oplus \nu_{X_i \subset W})}{e(\nu_{X_i \subset W})} \right).$$

The bundle  $\nu_{X_i \subset W}$  is trivial as a non-equivariant bundle, so the equivariant bundle  $TX_i \oplus \nu_{X_i \subset W}$  is isomorphic to  $TX_i \oplus \pi^* V_i$ . Thus

$$p(TX_i \oplus \nu_{X_i \subset W}) = \pi^* p(V_i) \otimes p(TX_i) = \pi^* p(V_i) \otimes 1 \in H_T^* \otimes H^*(X_i)$$

as  $TX_i$  is stably trivial. Hence  $\frac{p_I(TX_i \oplus \nu_{X_i \subset W})}{e(\nu_{X_i \subset W})} = \pi^* \left( \frac{p_I(V_i)}{e(V_i)} \right) \otimes 1$  which pushes forward to zero, so  $\kappa_{p_I} = 0$ .  $\square$

We now give our example.

**Example 4.5.** Let  $T = (S^1)^2$  and  $V_1$  be the 2-dimensional complex  $T$ -representation with weights  $\{x_1 + x_2, x_2\}$ , and  $V_2$  be the 2-dimensional complex  $T$ -representation with weights  $\{x_1, x_2\}$ . Construction 4.3 with  $B_1 = 2$  and  $B_2 = -2$  yields a  $T$ -manifold  $W$  (which may be chosen to have any dimension at least 6 and congruent to 2 modulo 4) having  $\kappa_{p_I} = 0$  and

$$\kappa_{ep_I} = 2(p_I(V_1) - p_I(V_2)) \in H_T^* = \mathbb{Q}[x_1, x_2].$$

For the chosen representations we have

$$\begin{aligned} p(V_1) &= (1 - (x_1 + x_2)^2)(1 - x_2^2) \\ p(V_2) &= (1 - x_1^2)(1 - x_2^2). \end{aligned}$$

Let us consider the image of the tautological subring  $R_T^* \leq H_T^* = \mathbb{Q}[x_1, x_2]$  in the quotient  $\mathbb{Q}[x_1, x_2]/(x_2^2)$ . Here  $p_2(V_1) = p_2(V_2) = 0$  and

$$p_1(V_1) = -(2x_1x_2 + x_1^2)$$

$$p_1(V_2) = -x_1^2,$$

so the only non-zero  $\kappa_{ep_I}$  in this quotient ring are

$$\kappa_{ep_1^i} = 2(-1)^i((2x_1x_2 + x_1^2)^i - (x_1^2)^i) = 4(-1)^i x_1^{2i-1} x_2,$$

so the image of  $R_T^*$  in  $\mathbb{Q}[x_1, x_2]/(x_2^2)$  is the subring  $S := \mathbb{Q}\langle x_1x_2, x_1^3x_2, x_1^5x_2, \dots \rangle$ . The ring  $S$  is an infinite-dimensional  $\mathbb{Q}$  vector space, as the  $x_1^{2i-1}x_2$  all have different degrees and are non-zero as they are not divisible by  $x_2^2$ . On the other hand, multiplication of any two positive-degree elements in  $S$  is zero, as each positive-degree element is divisible by  $x_2$  so a product is divisible by  $x_2^2$ . Thus  $S$  is infinitely-generated, so  $R_T^*$  is too, and hence  $R^*(W)/\sqrt{0}$  is too.

Let us record some observations about this example.

*Remark 4.6.* If we suppose that  $n \geq 5$  is odd and the  $T$ -manifolds  $M(2, V_1)$  and  $M(-2, V_2)$  are glued along a free orbit as suggested above, then the  $(2n+4)$ -manifold  $M$  obtained is simply-connected and has the same integral homology as

$$(S^2 \times S^{2n+2}) \# (S^2 \times S^{2n+2}) \# (S^3 \times S^{2n+1}) \# (S^n \times S^{n+4}) \# (S^n \times S^{n+4}).$$

*Remark 4.7.* Proposition 3.6 applies to this torus action and we nonetheless have the estimate  $\text{Kdim}(R^*(W)) \geq 2$ . (Specifically, we have

$$p_1(V_1) - p_1(V_2) = -(2x_1x_2 + x_2^2) \quad p_2(V_1) - p_2(V_2) = x_2^2(2x_1x_2 + x_2^2)$$

so after inverting  $s := x_2(2x_1 + x_2) \neq 0 \in R_T^*$  the subring  $s^{-1}R_T^* \leq s^{-1}H_T^*$  contains  $p_1(V_1), p_2(V_1), p_1(V_2)$ , and  $p_2(V_2)$ .)

*Remark 4.8.* Choosing  $* \in X_2$  gives a map  $R^*(W, *) \rightarrow H_T^*$ , whose image in  $\mathbb{Q}[x_1, x_2]/(x_2^2)$  is the subring  $S'$  generated by  $S$  along with  $x_1^2$ . But then  $S'$  is  $\mathbb{Q}\langle x_1x_2, x_1^2 \rangle$  so is finitely-generated. (Similarly if we choose  $* \in X_1$ .) This raises the interesting possibility that  $R^*(W, *)$  might be finitely-generated in more generality than  $R^*(W)$  is.

**4.4. The complex projective plane.** Let us consider the manifold  $\mathbb{CP}^2$ , whose cohomology is supported in even degrees. Thus by Theorem A the algebra  $R^*(\mathbb{CP}^2)$  is finitely-generated and  $R^*(\mathbb{CP}^2, *)$  is a finite  $R^*(\mathbb{CP}^2)$ -module.

The trace identity technique of Section 2.2.1 gives the relation

$$c^3 = \kappa_{ec}c^2 - \frac{\kappa_{ec}^2 - \kappa_{ec^2}}{2!}c + \frac{\kappa_{ec}^3 - 3\kappa_{ec}\kappa_{ec^2} + 2\kappa_{ec^3}}{3!} \in R^*(\mathbb{CP}^2, *)$$

for any  $c \in H^*(BSO(4)) = \mathbb{Q}[p_1, e]$ . In particular, for  $c = e$  and  $c = p_1$  we obtain

$$(4.1) \quad e^3 = \kappa_{e^2}e^2 - \frac{\kappa_{e^2}^2 - \kappa_{e^3}}{2!}e + \frac{\kappa_{e^2}^3 - 3\kappa_{e^2}\kappa_{e^3} + 2\kappa_{e^4}}{3!}$$

$$(4.2) \quad p_1^3 = \kappa_{ep_1}p_1^2 - \frac{\kappa_{ep_1}^2 - \kappa_{ep_1^2}}{2!}p_1 + \frac{\kappa_{ep_1}^3 - 3\kappa_{ep_1}\kappa_{ep_1^2} + 2\kappa_{ep_1^3}}{3!}$$

We may polarise the relation by taking  $c = e + A \cdot p_1$ , expanding and collecting coefficients of powers of  $A$ . The coefficients of 1 and of  $A^3$  simply give the relations (4.1) and (4.2). The coefficient of  $A$  gives

$$(4.3) \quad \begin{aligned} & -\kappa_{e^3p_1} - (1/2)\kappa_{e^2}^2\kappa_{ep_1} - \kappa_{e^2p_1}e - \kappa_{ep_1}e^2 + (1/2)\kappa_{e^2}^2p_1 \\ & + \kappa_{e^2}\kappa_{e^2p_1} - (1/2)\kappa_{e^3}p_1 + (1/2)\kappa_{ep_1}\kappa_{e^3} + \kappa_{e^2}\kappa_{ep_1}e - 2\kappa_{e^2}ep_1 + 3e^2p_1 = 0 \end{aligned}$$

and the coefficient of  $A^2$  gives

$$(4.4) \quad \begin{aligned} & (1/2)e\kappa_{ep_1}^2 - (1/2)\kappa_{ep_1}^2\kappa_{e^2} - \kappa_{e^2p_1^2} - (1/2)e\kappa_{ep_1^2} + \kappa_{ep_1}\kappa_{e^2p_1} \\ & + (1/2)\kappa_{ep_1^2}\kappa_{e^2} - p_1\kappa_{e^2p_1} - p_1^2\kappa_{e^2} + p_1\kappa_{ep_1}\kappa_{e^2} - 2ep_1\kappa_{ep_1} + 3ep_1^2 = 0. \end{aligned}$$

The relations (4.1), (4.2), (4.3) and (4.4), multiplied by monomials in  $\mathbb{Q}[p_1, e]$  and pushed forward, show that certain  $\kappa_{e^a p_1^b} \in R^*(\mathbb{CP}^2)$  are decomposable. Specifically

$$\begin{aligned} \kappa_{xp_1^3} &\text{ is decomposable for any monomial } x \neq 1, e, p_1 \\ \kappa_{xep_1^2} &\text{ is decomposable for any monomial } x \neq 1, e, p_1 \\ \kappa_{xe^2 p_1} &\text{ is decomposable for any monomial } x \neq 1, e, p_1 \\ \kappa_{xe^3} &\text{ is decomposable for any monomial } x \neq 1, e, p_1. \end{aligned}$$

Writing  $\equiv$  to mean “equal modulo decomposables”, there are further relations:

- (i) Pushing (4.2) forward gives  $\kappa_{p_1^3} \equiv \frac{3}{2}\kappa_{ep_1^2}$ .
- (ii) Pushing (4.2) multiplied by  $p_1$  forward gives  $\kappa_{p_1^4} \equiv \kappa_{ep_1^3}$ .
- (iii) Pushing (4.1) forward gives that  $\kappa_{e^3}$  is decomposable, and in fact that  $\kappa_{e^3} = \kappa_{e^2}^2$ .
- (iv) Pushing (4.1) multiplied by  $p_1$  forward gives  $\kappa_{e^3 p_1} \equiv \kappa_{e^4}$ .
- (v) Pushing (4.3) multiplied by  $p_1$  forward gives  $\kappa_{e^2 p_1^2} \equiv \kappa_{e^3 p_1}$ .
- (vi) Pushing (4.4) forward gives  $2\kappa_{e^2 p_1} \equiv -\kappa_{ep_1^2}$ .
- (vii) Pushing (4.4) multiplied by  $p_1$  forward gives  $\kappa_{ep_1^3} \equiv \kappa_{e^2 p_1^2}$ .

Using these relations we find that the five classes

$$\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{p_1^4}, \kappa_{ep_1}, \kappa_{e^2} \in R^*(\mathbb{CP}^2)$$

generate.

As described in [GGR15, Section 2], it follows from work of Atiyah [Ati69] that for each Hirzebruch class  $\mathcal{L}_i$  the associated class  $\kappa_{\mathcal{L}_i} \in R^*(W)$  is pulled back via the natural map

$$\phi : B\text{Diff}^+(W) \longrightarrow B\text{Aut}(H, \lambda),$$

where  $H = H^n(W; \mathbb{Z})/\text{torsion}$  and  $\lambda : H \otimes H \rightarrow \mathbb{Z}$  is the intersection form of  $W$ .

For  $W = \mathbb{CP}^2$  the bilinear form  $(H, \lambda) = (\mathbb{Z}, (1))$  has automorphism group  $\mathbb{Z}/2$ , which has trivial rational cohomology. Thus the classes  $\kappa_{\mathcal{L}_i} \in R^*(\mathbb{CP}^2)$  are zero. The first few are

$$\begin{aligned} 7\kappa_{e^2} - \kappa_{p_1^2} &= 0 \\ -13\kappa_{e^2 p_1} + 2\kappa_{p_1^3} &= 0 \\ -19\kappa_{e^4} + 22\kappa_{e^2 p_1^2} - 3\kappa_{p_1^4} &= 0 \\ 127\kappa_{e^4 p_1} - 83\kappa_{e^2 p_1^3} + 10\kappa_{p_1^5} &= 0 \\ 8718\kappa_{e^6} - 27635\kappa_{e^4 p_1^2} + 12842\kappa_{e^2 p_1^4} - 1382\kappa_{p_1^6} &= 0 \\ -7978\kappa_{e^6 p_1} + 11880\kappa_{e^4 p_1^3} - 4322\kappa_{e^2 p_1^5} + 420\kappa_{p_1^7} &= 0 \\ -68435\kappa_{e^8} + 423040\kappa_{e^6 p_1^2} - 407726\kappa_{e^4 p_1^4} + 122508\kappa_{e^2 p_1^6} - 10851\kappa_{p_1^8} &= 0 \\ 11098737\kappa_{e^8 p_1} - 29509334\kappa_{e^6 p_1^3} + 20996751\kappa_{e^4 p_1^5} - 5391213\kappa_{e^2 p_1^7} + 438670\kappa_{p_1^9} &= 0. \end{aligned}$$

The first Hirzebruch relation allows up to remove  $\kappa_{e^2}$  from the list of generators. The second Hirzebruch relation, with the relations  $\kappa_{p_1^3} \equiv \frac{3}{2}\kappa_{ep_1^2} \equiv -3\kappa_{e^2 p_1}$  proved above, shows that  $\kappa_{p_1^3}$  is decomposable. This proves the

**Lemma 4.9.** *The classes  $\kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}$  generate  $R^*(\mathbb{CP}^2)$ .*

The ideal  $I \triangleleft \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]$  of relations implied by (4.1)–(4.4) for  $\kappa_{e^a p_1^b}$  for  $a + b \leq 9$ , and the Hirzebruch relations listed above, is generated by

$$\begin{aligned} (4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})\kappa_{p_1^4} \\ (4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})(21\kappa_{ep_1} + 8\kappa_{p_1^2}) \\ (4\kappa_{p_1^2} - 7\kappa_{ep_1})(316\kappa_{ep_1}^3 - 343\kappa_{p_1^4}) \\ (4\kappa_{p_1^2} - 7\kappa_{ep_1})(1264\kappa_{p_1^2}\kappa_{ep_1}^2 + 2212\kappa_{ep_1}^3 - 5145\kappa_{p_1^4}). \end{aligned}$$



This ideal is not radical, and  $\sqrt{I}$  is generated by

$$\begin{aligned} & (4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1}) \\ & (4\kappa_{p_1^2} - 7\kappa_{ep_1})(316\kappa_{ep_1}^3 - 343\kappa_{p_1^4}). \end{aligned}$$

**Corollary 4.10.** *There is a surjection from*

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}] / ((4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1}), (4\kappa_{p_1^2} - 7\kappa_{ep_1})(316\kappa_{ep_1}^3 - 343\kappa_{p_1^4}))$$

to  $R^*(\mathbb{CP}^2)/\sqrt{0}$ , and hence  $\text{Kdim}(R^*(\mathbb{CP}^2)) \leq 2$ .

Thus there is a surjection  $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/I \rightarrow R^*(\mathbb{CP}^2)$  and hence a surjection  $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/\sqrt{I} \rightarrow R^*(\mathbb{CP}^2)/\sqrt{0}$ . In particular  $\text{Kdim}(R^*(\mathbb{CP}^2)) \leq 2$ .

4.4.1. *Fixing a point.* It follows from Lemma 4.9 that  $R^*(\mathbb{CP}^2, *)$  is generated by  $e, p_1, \kappa_{p_1^2}, \kappa_{p_1^4}$  and  $\kappa_{ep_1}$ . Adding to the ideal  $I$  above the relations (4.1)–(4.4) gives an ideal  $J \triangleleft \mathbb{Q}[e, p_1, \kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]$  which is rather complicated, but its radical is generated by the relations

$$\begin{aligned} & (4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1}) \\ & 1264\kappa_{p_1^2}^3\kappa_{ep_1} - 2212\kappa_{ep_1}^4 - 1372\kappa_{p_1^2}\kappa_{p_1^4} + 2401\kappa_{p_1^4}\kappa_{ep_1} \\ & 10\kappa_{p_1^2}\kappa_{ep_1} - 28\kappa_{p_1^2}p_1 - 21\kappa_{ep_1}^2 - 14\kappa_{ep_1}e + 63\kappa_{ep_1}p_1 \\ & 3\kappa_{p_1^2}\kappa_{ep_1} - 28\kappa_{p_1^2}e - 7\kappa_{ep_1}^2 + 42\kappa_{ep_1}e + 7\kappa_{ep_1}p_1 \\ & 45\kappa_{p_1^2}\kappa_{ep_1} - 112\kappa_{ep_1}^2 - 84\kappa_{ep_1}e + 182\kappa_{ep_1}p_1 + 196e^2 - 196p_1^2 \\ & 15\kappa_{p_1^2}\kappa_{ep_1} - 35\kappa_{ep_1}^2 + 14\kappa_{ep_1}e + 35\kappa_{ep_1}p_1 + 196e^2 - 196ep_1 \\ & 316\kappa_{ep_1}^4 + 1264\kappa_{ep_1}^3e - 1264\kappa_{ep_1}^3p_1 - 343\kappa_{p_1^4}\kappa_{ep_1} - 1372\kappa_{p_1^4}e + 1372\kappa_{p_1^4}p_1 \\ & 12263\kappa_{p_1^2}\kappa_{ep_1}^2 - 19446\kappa_{ep_1}^3 + 168\kappa_{ep_1}^2e - 168\kappa_{ep_1}^2p_1 - 4116\kappa_{ep_1}e^2 + 16464e^3 - 5488\kappa_{p_1^4} \end{aligned}$$

the last of which shows that the generator  $\kappa_{p_1^4}$  may be eliminated from the ring  $\mathbb{Q}[e, p_1, \kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]/\sqrt{J}$ . One may also deduce from these relations that  $\kappa_{ep_1}$  and  $\kappa_{p_1^2}$  are integral over  $\mathbb{Q}[e, p_1]$ , so that  $R^*(\mathbb{CP}^2, *)/\sqrt{0}$  is finite over  $\mathbb{Q}[e, p_1]$ .

4.4.2. *Fixing a disc.* As passing from  $R^*(\mathbb{CP}^2, *)$  to  $R^*(\mathbb{CP}^2, D^4)$  in particular kills  $e$  and  $p_1$ , we deduce from the above that

**Corollary 4.11.**  *$R^*(\mathbb{CP}^2, D^4)$  is a finite-dimensional  $\mathbb{Q}$ -vector space.*

In fact, setting  $K = J + (e, p_1)$  and simplifying, we find that  $K$  is generated by

$$\begin{aligned} & \kappa_{p_1^4}^2 & \kappa_{p_1^2}\kappa_{p_1^4} \\ & \kappa_{ep_1}\kappa_{p_1^4} & 29\kappa_{p_1^2}^3 - 245\kappa_{p_1^4} \\ & 609\kappa_{ep_1}^3 - 260\kappa_{p_1^4} & 29\kappa_{p_1^2}\kappa_{ep_1}^2 - 52\kappa_{p_1^4} \\ & 87\kappa_{p_1^2}^2\kappa_{ep_1} - 77\kappa_{p_1^4} \end{aligned}$$

and  $R^*(\mathbb{CP}^2, D^4)$  is a quotient of  $\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^4}, \kappa_{ep_1}]/K$  so

**Corollary 4.12.**  $\dim_{\mathbb{Q}} R^*(\mathbb{CP}^2, D^4) \leq 7$ .

4.4.3. *Lower bounds via torus actions.* Consider the toric action of  $T = (S^1)^2$  on  $\mathbb{CP}^2$ , giving a map

$$\phi : R^*(\mathbb{CP}^2) \longrightarrow H_T^* = \mathbb{Q}[x_1, x_2].$$

It is an elementary exercise to compute, by equivariant localisation, the classes

$$\begin{aligned} \phi(\kappa_{p_1^2}) &= 7x_1^2 - 7x_1x_2 + 7x_2^2 \\ \phi(\kappa_{ep_1}) &= 4x_1^2 - 4x_1x_2 + 4x_2^2 \end{aligned}$$

$$\phi(\kappa_{p_1^4}) = 23x_1^6 - 69x_1^5x_2 + 135x_1^4x_2^2 - 155x_1^3x_2^3 + 135x_1^2x_2^4 - 69x_1x_2^5 + 23x_2^6$$

and by eliminating variables to find that the unique relation between these is  $\phi(7\kappa_{ep_1} - 4\kappa_{p_1^2}) = 0$ . Thus  $\phi$  gives a surjection

$$R^*(\mathbb{CP}^2)/\sqrt{0} \longrightarrow \mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/(7\kappa_{ep_1} - 4\kappa_{p_1^2})$$

and hence  $\text{Kdim}(R^*(\mathbb{CP}^2)) \geq 2$ . Combining this with the above gives

**Corollary 4.13.**  $\text{Kdim}(R^*(\mathbb{CP}^2)) = 2$ .

Choosing a fixed point of the  $T$ -action gives an extension of  $\phi$  to a

$$\hat{\phi}: R^*(\mathbb{CP}^2, *)/\sqrt{0} \longrightarrow H_T^* = \mathbb{Q}[x_1, x_2].$$

For a particular choice of fixed point we have

$$\begin{aligned}\hat{\phi}(s^*e) &= x_1x_2 \\ \hat{\phi}(s^*p_1) &= x_1^2 + x_2^2\end{aligned}$$

which shows that the image of  $\hat{\phi}$  is isomorphic to

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}, e, p_1]/(\kappa_{p_1^2} - 7p_1 + 7e, \kappa_{ep_1} - 4p_1 + 4e, 17e^3 - 66e^2p_1 + 69ep_1^2 - 23p_1^3 + \kappa_{p_1^4}),$$

or in other words  $\mathbb{Q}[e, p_1]$ .

*Remark 4.14.* In [Fin77, Fin78] there is given an analysis of  $S^1$ -actions on simply-connected 4-manifolds, from which it is possible to deduce—through a very laborious consideration of cases and analysis of fixed-point data—that for any circle action on  $\mathbb{CP}^2$  we have  $4\kappa_{e^2} = \kappa_{ep_1}$  and so by the first Hirzebruch relation we have  $4\kappa_{p_1^2} - 7\kappa_{ep_1} = 0$ . Alternatively, this may be proved using Hsiang's splitting theorem for the  $S^1$ -equivariant cohomology of  $\mathbb{CP}^2$  [Hsi75, Theorem VI.1].

**4.4.4. The tautological variety.** We find it quite revealing to consider the (reduced) tautological ring  $R^*(\mathbb{CP}^2)/\sqrt{0}$  by considering its associated variety  $\mathbf{V}_{\mathbb{CP}^2}$ . The choice of generators  $\kappa_{p_1^2}$ ,  $\kappa_{p_1^4}$ , and  $\kappa_{ep_1}$  of  $R^*(\mathbb{CP}^2)$  presents  $\mathbf{V}_{\mathbb{CP}^2}$  as a subvariety of  $\mathbf{A}^3$ , and it follows from Corollary 4.10 that  $\mathbf{V}_{\mathbb{CP}^2}$  is contained in the union of the plane

$$\mathbf{P} := \{4\kappa_{p_1^2} - 7\kappa_{ep_1} = 0\}$$

and the line

$$\mathbf{L} := \{\kappa_{p_1^2} - 2\kappa_{ep_1} = 0, 316\kappa_{ep_1}^3 - 343\kappa_{p_1^4} = 0\}.$$

Furthermore, it follows from the calculation of Section 4.4.3 that  $\mathbf{V}_{\mathbb{CP}^2}$  contains  $\mathbf{P}$ , so the variety  $\mathbf{V}_{\mathbb{CP}^2}$  is either  $\mathbf{P}$  or  $\mathbf{P} \cup \mathbf{L}$ . It would be extremely interesting if  $\mathbf{L} \subset \mathbf{V}_{\mathbb{CP}^2}$ , but no method for showing this seems to be available. (Each circle action on  $\mathbb{CP}^2$  gives a homomorphism  $R^*(\mathbb{CP}^2)/\sqrt{0} \rightarrow \mathbb{Q}[x_1]$  and hence a morphism  $\mathbf{A}^1 \rightarrow \mathbf{V}_{\mathbb{CP}^2}$ , but by Remark 4.14 all such morphisms have image in  $\mathbf{P}$ .)

Similarly, by the calculation of Section 4.4.1 the four elements  $e, p_1, \kappa_{p_1^2}$ , and  $\kappa_{ep_1}$  generate  $R^*(\mathbb{CP}^2, *)/\sqrt{0}$ , which presents the associated variety  $\mathbf{V}_{(\mathbb{CP}^2, *)}$  as a subvariety of  $\mathbf{A}^4$ . Eliminating the variable  $\kappa_{p_1^4}$  from the radical ideal described in Section 4.4.1 shows that  $\mathbf{V}_{(\mathbb{CP}^2, *)}$  is contained in the union of the plane

$$\{4\kappa_{p_1^2} - 7\kappa_{ep_1} = 0, \kappa_{ep_1} - 4p_1 + 4e = 0\}$$

the line

$$\{\kappa_{p_1^2} - 2\kappa_{ep_1} = 0, e = 0, \kappa_{ep_1} - 7p_1 = 0\}$$

and the line

$$\{\kappa_{p_1^2} - 2\kappa_{ep_1} = 0, 2\kappa_{ep_1} - 7e = 0, 5\kappa_{ep_1} - 7p_1 = 0\}.$$

It follows from the calculation of Section 4.4.3 that the plane is contained in  $\mathbf{V}_{(\mathbb{CP}^2, *)}$ .

4.5. **The manifold  $S^2 \times S^2$ .** The cohomology of  $S^2 \times S^2$  is supported in even degrees. Thus by Theorem A the algebra  $R^*(S^2 \times S^2)$  is finitely-generated and  $R^*(S^2 \times S^2, *)$  is a finite  $R^*(S^2 \times S^2)$ -module.

The trace identity technique of Section 2.2.1 gives the relation

$$c^4 = \kappa_{ec}c^3 - \frac{\kappa_{ec}^2 - \kappa_{ec^2}}{2}c^2 + \frac{\kappa_{ec}^3 + 3\kappa_{ec}\kappa_{ec^2} - 2\kappa_{ec^3}}{6}c - \frac{\kappa_{ec}^4}{24} + \frac{\kappa_{ec}^2\kappa_{ec^2}}{4} - \frac{\kappa_{ec^2}^2}{8} - \frac{\kappa_{ec}\kappa_{ec^3}}{3} + \frac{\kappa_{ec^4}}{4} \in R^*(S^2 \times S^2, *)$$

for any  $c \in H^*(BSO(4)) = \mathbb{Q}[p_1, e]$ . Polarising via  $c = e + A \cdot p_1$  as in Section 4.4, we obtain the relations

$$(4.5) \quad \begin{aligned} & (1/8)\kappa_{ep_1^2}^2 + (1/24)\kappa_{ep_1}^4 - (1/6)p_1\kappa_{ep_1}^3 - (1/4)\kappa_{ep_1}^2\kappa_{ep_1^2} \\ & + (1/2)p_1^2\kappa_{ep_1}^2 - (1/3)p_1\kappa_{ep_1^3} + (1/3)\kappa_{ep_1}\kappa_{ep_1^3} - p_1^3\kappa_{ep_1} \\ & - (1/2)p_1^2\kappa_{ep_1^2} - (1/4)\kappa_{ep_1^4} + p_1^4 + (1/2)p_1\kappa_{ep_1}\kappa_{ep_1^2} = 0 \end{aligned}$$

$$(4.6) \quad \begin{aligned} & - p_1^2\kappa_{e^2p_1} - (1/6)\kappa_{ep_1}^3e + \kappa_{ep_1}\kappa_{e^2p_1^2} - p_1\kappa_{e^2p_1^2} + (1/6)\kappa_{ep_1}^3\kappa_{e^2} \\ & + (1/3)\kappa_{ep_1^3}\kappa_{e^2} - p_1^3\kappa_{e^2} + (1/2)\kappa_{ep_1^2}\kappa_{e^2p_1} - (1/2)\kappa_{ep_1}^2\kappa_{e^2p_1} \\ & + 4ep_1^3 - (1/3)e\kappa_{ep_1^3} - (1/2)\kappa_{ep_1}\kappa_{ep_1^2}\kappa_{e^2} + (1/2)p_1\kappa_{ep_1^2}\kappa_{e^2} \\ & - (1/2)p_1\kappa_{ep_1}^2\kappa_{e^2} + p_1^2\kappa_{ep_1}\kappa_{e^2} - ep_1\kappa_{ep_1^2} + (1/2)e\kappa_{ep_1}\kappa_{ep_1^2} \\ & + ep_1\kappa_{ep_1}^2 - 3ep_1^2\kappa_{ep_1} + p_1\kappa_{ep_1}\kappa_{e^2p_1} - \kappa_{e^2p_1^3} = 0 \\ & 6e^2p_1^2 + (1/4)\kappa_{ep_1}^2\kappa_{e^2}^2 + (1/2)p_1^2\kappa_{e^2}^2 - (1/2)e^2\kappa_{ep_1^2} \\ & + (1/2)e^2\kappa_{ep_1}^2 - p_1\kappa_{e^3p_1} + \kappa_{ep_1}\kappa_{e^3p_1} + (1/4)\kappa_{ep_1^2}\kappa_{e^3} - (1/4)\kappa_{ep_1}^2\kappa_{e^3} \\ & - (1/2)p_1^2\kappa_{e^3} - e\kappa_{e^2p_1^2} + \kappa_{e^2}\kappa_{e^2p_1^2} - (1/4)\kappa_{ep_1^2}\kappa_{e^2}^2 \\ & - (1/2)e\kappa_{ep_1}^2\kappa_{e^2} - 3ep_1^2\kappa_{e^2} - 3e^2p_1\kappa_{ep_1} \\ & + (1/2)p_1\kappa_{ep_1}\kappa_{e^3} + (1/2)\kappa_{e^2p_1}^2 + (1/2)e\kappa_{ep_1^2}\kappa_{e^2} \\ & - (1/2)p_1\kappa_{ep_1}\kappa_{e^2}^2 + e\kappa_{ep_1}\kappa_{e^2p_1} + p_1\kappa_{e^2}\kappa_{e^2p_1} \\ & - 2ep_1\kappa_{e^2p_1} - \kappa_{ep_1}\kappa_{e^2}\kappa_{e^2p_1} + 2ep_1\kappa_{ep_1}\kappa_{e^2} - (3/2)\kappa_{e^3p_1^2} = 0 \end{aligned}$$

$$(4.7) \quad \begin{aligned} & (1/2)\kappa_{e^2p_1}\kappa_{e^3} - (1/2)\kappa_{e^2}\kappa_{e^2p_1} - e^2\kappa_{e^2p_1} - (1/6)\kappa_{e^3}^3p_1 \\ & + (1/6)\kappa_{e^3}^3\kappa_{ep_1} + \kappa_{e^2}\kappa_{e^3p_1} - e^3\kappa_{ep_1} + (1/3)\kappa_{ep_1}\kappa_{e^4} - (1/3)p_1\kappa_{e^4} \\ & - e\kappa_{e^3p_1} + (1/2)p_1\kappa_{e^2}\kappa_{e^3} - (1/2)\kappa_{ep_1}\kappa_{e^2}\kappa_{e^3} + (1/2)e\kappa_{ep_1}\kappa_{e^3} \\ & - ep_1\kappa_{e^3} + e\kappa_{e^2}\kappa_{e^2p_1} + ep_1\kappa_{e^2}^2 - (1/2)e\kappa_{ep_1}\kappa_{e^2}^2 - 3e^2p_1\kappa_{e^2} \\ & + e^2\kappa_{ep_1}\kappa_{e^2} - \kappa_{e^4p_1} + 4e^3p_1 = 0 \end{aligned}$$

$$(4.8) \quad \begin{aligned} & (1/8)\kappa_{e^3}^2 + (1/24)\kappa_{e^2}^4 - (1/6)e\kappa_{e^2}^3 + (1/2)e^2\kappa_{e^2}^2 - (1/4)\kappa_{e^2}^2\kappa_{e^3} \\ & + (1/2)e\kappa_{e^2}\kappa_{e^3} - (1/4)\kappa_{e^5} - e^3\kappa_{e^2} - (1/2)e^2\kappa_{e^3} - (1/3)e\kappa_{e^4} \\ & + (1/3)\kappa_{e^2}\kappa_{e^4} + e^4 = 0 \end{aligned}$$

Modulo decomposables in  $R^*(S^2 \times S^2)$ , these fibre integrate to give the relations

$$\begin{aligned} \kappa_{xp_1^4} &\equiv 0 \text{ for any monomial } x \neq e \in \mathbb{Q}[e, p_1] \\ \kappa_{xep_1^3} &\equiv 0 \text{ for any } x \in \mathbb{Q}[e, p_1] \\ \kappa_{xe^2p_1^2} &\equiv 0 \text{ for any } x \in \mathbb{Q}[e, p_1] \\ \kappa_{xe^3p_1} &\equiv 0 \text{ for any monomial } x \neq 1 \in \mathbb{Q}[e, p_1] \\ \kappa_{xe^4} &\equiv 0 \text{ for any monomial } x \neq e \in \mathbb{Q}[e, p_1]. \end{aligned}$$

Furthermore, multiplying the second relation above by  $p_1$  and fibre integrating shows that  $\kappa_{ep_1^4}$  is decomposable. This shows that all elements apart from

$$\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^2}, \kappa_{e^2 p_1}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}$$

are decomposable. The Hirzebruch relations of the previous section hold here as well, as the bilinear form associated to  $S^2 \times S^2$  is  $(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$  which also has finite automorphism group. The first two Hirzebruch relations  $\kappa_{e^2} = \frac{1}{7}\kappa_{p_1^2}$  and  $\kappa_{e^2 p_1} = \frac{2}{13}\kappa_{p_1^3}$  allow us to remove two of these elements, and so we find that

**Lemma 4.15.** *The classes  $\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}$  generate  $R^*(S^2 \times S^2)$ .*

Consider the ideal  $I \triangleleft \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]$  of relations implied by (4.5)–(4.9) for  $\kappa_{e^a p_1^b}$  for  $a + b \leq 9$ , and the Hirzebruch relations of the previous section. It is quite complicated, but it is easy to compute (in `Macaulay2`) that it has codimension 3.

**Corollary 4.16.**  $\text{Kdim}(R^*(S^2 \times S^2)) \leq 4$ .

4.5.1. *Fixing a point.* It follows from Lemma 4.9 that  $R^*(S^2 \times S^2, *)$  is generated by  $e, p_1, \kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}$ . Adding to the ideal  $I$  above the relations (4.5)–(4.9) gives an ideal  $J \triangleleft \mathbb{Q}[e, p_1, \kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}]$  which has codimension 5.

4.5.2. *Fixing a disc.* Passing from  $R^*(S^2 \times S^2, *)$  to  $R^*(S^2 \times S^2, D^4)$  in particular kills  $e$  and  $p_1$ , and we may compute the radical of the ideal  $K := J + (e, p_1)$ , giving the following.

**Corollary 4.17.**  $R^*(S^2 \times S^2, D^4)/\sqrt{0}$  is a quotient of

$$\mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^5}] / (\kappa_{p_1^3}, \kappa_{p_1^2}, \kappa_{e^3}^2 - 2\kappa_{e^5}, \kappa_{ep_1} \kappa_{e^3} - 2\kappa_{e^3 p_1}, \kappa_{ep_1}^2 - 2\kappa_{ep_1^2})$$

so has Krull dimension at most 2.

4.5.3. *Lower bounds via torus actions.* For each  $k$  there is an almost-complex torus action  $\phi_k : T^2 \rightarrow \text{Diff}(S^2 \times S^2)$  which has four fixed points, with weights

$$\{-x_2, x_1 - 2kx_2\}, \{-x_2, 2kx_2 - x_1\}, \{x_2, x_1\}, \{x_2, -x_1\}.$$

These actions seem to be well-known among symplectic geometers: we learnt their construction from work of Karshon [Kar03], suggested to us by Ivan Smith. In that paper these actions are constructed from their Delzant polytopes, from which the weights at the fixed points are immediately read off.

These actions have Euler class

$$x_1(2kx_1 - x_2), x_1(x_2 - 2kx_1), x_1x_2, -x_1x_2$$

and Pontrjagin class  $p_1 = c_1^2 - 2c_2$  equal to

$$(4k^2 + 1)x_1^2 - 4kx_2x_1 + x_2^2, (4k^2 + 1)x_1^2 - 4kx_2x_1 + x_2^2, x_1^2 + x_2^2, x_1^2 + x_2^2.$$

We may thus compute the map

$$\psi_k : \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3 p_1}, \kappa_{e^5}] \longrightarrow R^*(S^2 \times S^2) \xrightarrow{\phi_k} H_T^* = \mathbb{Q}[s, t]$$

by equivariant localisation, giving

$$\kappa_{p_1^2} = 0$$

$$\kappa_{p_1^3} = 0$$

$$\kappa_{ep_1} = 8k^2x_1^2 - 8kx_1x_2 + 4x_1^2 + 4x_2^2$$

$$\begin{aligned} \kappa_{ep_1^2} = & 32k^4x_1^4 - 64k^3x_1^3x_2 + 16k^2x_1^4 + 48k^2x_1^2x_2^2 - 16kx_1^3x_2 - 16kx_1x_2^3 + 4x_1^4 \\ & + 8x_1^2x_2^2 + 4x_2^4 \end{aligned}$$

$$\begin{aligned}
\kappa_{e^3} &= 8k^2x_1^4 - 8kx_1^3x_2 + 4x_1^2x_2^2 \\
\kappa_{e^3p_1} &= 32k^4x_1^6 - 64k^3x_1^5x_2 + 8k^2x_1^6 + 48k^2x_1^4x_2^2 - 8kx_1^5x_2 - 16kx_1^3x_2^3 + 4x_1^4x_2^2 \\
&\quad + 4x_1^2x_2^4 \\
\kappa_{e^5} &= 32k^4x_1^8 - 64k^3x_1^7x_2 + 48k^2x_1^6x_2^2 - 16kx_1^5x_2^3 + 4x_1^4x_2^4
\end{aligned}$$

By eliminating  $x_1$ ,  $x_2$ , and  $k$  from the above, one finds generators for the ideal  $U := \bigcap_{k \in \mathbb{N}} \text{Ker}(\psi_k) \triangleleft \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3p_1}, \kappa_{e^5}]$ . Thus there is a surjection

$$R^*(S^2 \times S^2) \longrightarrow \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3p_1}, \kappa_{e^5}] / U,$$

and hence the Krull dimension of  $R^*(S^2 \times S^2)$  is bounded below by the codimension of the ideal  $U$ . This is easily computed to be 3.

**Corollary 4.18.**  $\text{Kdim}(R^*(S^2 \times S^2)) \geq 3$ .

Note that each ideal  $\text{Ker}(\psi_k) \triangleleft \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3p_1}, \kappa_{e^5}]$  has codimension 2, so each particular torus action only gives 2 as a lower bound for  $\text{Kdim}(R^*(S^2 \times S^2))$ : it is only by considering the countably-many such actions that we are able to improve this lower bound to 3.

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